

GEOMETRIC DESCRIPTION OF THE THERMODYNAMICS OF A BLACK HOLE WITH POWER MAXWELL INVARIANT SOURCE

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Abstract

Considering a nonlinear charged black hole as a thermodynamics system, we study the geometric description of its phase transitions. Using the formalism of geometrothermodynamics we show that the geometry of the space of thermodynamic equilibrium states of this kind of black holes is related with information about thermodynamic interaction, critical points and phase transitions structure. Our results indicate that the equilibrium manifold of this black hole is curved and that curvature singularities appear exactly at those places where first and second order phase transitions occur.

Keywords: Thermodynamic, Phase transition, black holes

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I. INTRODUCTION

AdS spacetimes in $n+1$ -dimensions has won notoriety since the norm/gravity correspondence was postulated in 1997 by Maldacena [1–3] and black holes in the gravity side are related with the temperature of the system in the field theory side. However the thermodynamic study of black holes in AdS spacetime was initiated long before the duality AdS/CFT appeared by Hawking and Page [4] as a curiosity in theoretical physics. They show that the Schwarzschild-AdS black hole has a phase transition and this quality attracted others to keep studying this objects in AdS backgrounds. The Reissner-Nordström-AdS (RNAdS) black hole in $n+1$ -dimensions has been studied in [5] where they found that also RNAdS present a phase transition. Later, some authors considered the cosmological constant as a new thermodynamical variable in AdS black holes [6, 7], extending in this way the phase space. Recently, nonlinear electrodynamics source in AdS-black holes rise interest in the community, but focusing in particular in the power Maxwell invariant field (PMI) [8–10] where a s parameter is taken as a power of the Maxwell term in the action, i.e. $(F_{\mu\nu}F^{\mu\nu})^s$, which reduces to the Maxwell field (linear electromagnetic source) when $s = 1$. In those theories the authors found that black holes with nonlinear source present a transition phase in both, the canonical ensemble and the grand canonical ensemble.

In order to describe the behaviour of the thermodynamic system by means of geometry, different approaches have been proposed [11–14] which use a Riemannian manifold to define a space of equilibrium states where the thermodynamic phenomena take place. These approaches introduce a metric by means of which the thermodynamic concepts are related to the geometry.

This work is organized as follows. In section II, we present the elements of the geometrothermodynamics formalism that will be used in this work to obtain the thermodynamical behaviour of the systems analyzed. In section III, we present the black hole solution in AdS spacetime with power invariant Maxwell source (PMI) and the thermodynamic attributes of this system. In section IV, we apply the geometrothermodynamics (GTD) formalism to black holes with PMI source. In the section V, we consider the case of the Reissner-Nordström black hole where; its phase transitions are studied and we show that the Legendre-invariant metric reproduces its corresponding phase transition structure. Section VI, contains the results of analyzing the thermodynamics of the black holes with

PMI source solution using the Weinhold geometry. In section VII, we analyze the thermodynamics behaviour of a black hole with PMI source considering the cosmological constant as a thermodynamical variable under GTD. Finally, in section VIII, we present our conclusions.

II. GEOMETROTHERMODYNAMICS APPROACH

Geometrothermodynamics (GTD) [14, 15] is a geometric formalism associated to physical systems in order to obtain its thermodynamical behaviour. The first element to consider under this formalism is a contact manifold \mathcal{T} of dimension $(2n+1)$, called phase space, with a metric G defined on it and a contact 1-form Θ that satisfies the condition $\Theta \wedge (d\Theta)^n \neq 0$. On this geometrical structure, (\mathcal{T}, G, Θ) , can be use a set of coordinates $\{Z^A\}_{A=1,\dots,2n+1} \equiv \{\Phi, E^a, I^a\}_{a=1,\dots,n}$ and demand G to be invariant by a total Legendre transformation. This means that when it is performing a change of coordinates $\{Z^A\}_{A=1,\dots,2n+1}$ to $\{\tilde{Z}^A\}_{A=1,\dots,2n+1}$ that satisfies the transformations

$$\Phi = \tilde{\Phi} - \tilde{E}_a \tilde{I}^a, \quad E^a = -\tilde{I}^a, \quad I^a = \tilde{E}^a, \quad (1)$$

then the new metric \tilde{G} has the same structure as G defined over its own set of coordinates.

In GTD, we can also define an embedding $\varphi : \mathcal{E} \longrightarrow \mathcal{T}$ as $\varphi : (E^a) \longrightarrow (\Phi, E^a, I^a)$ with $\Phi(E^a)$ such that $\varphi^*(\Theta) = \varphi^*(d\Phi - \delta_{ab} I^a dE^b) = 0$ holds. The n -dimensional Riemannian submanifold \mathcal{E} is called the thermodynamic equilibrium space, Φ is called the thermodynamic potential, E^a are called extensive variables and I^a are called intensive variables. With these geometric elements, all the thermodynamic information of the system can be extracted by using the mapping φ and the condition of the pullback acting on Θ .

A GTD metric G that satisfies the previous requirements and that we will be using in this work is given by

$$G = \Theta^2 + (\delta_{ab} E^a I^b)(\eta_{cd} dE^c dI^d), \quad (2)$$

where $\delta_{ab} = \text{diag}(1, 1, \dots, 1)$, $\eta_{ab} = \text{diag}(-1, 1, \dots, 1)$ and $\Theta = d\Phi - \delta_{ab} I^a dE^b$ and

$$\frac{\partial \Phi}{\partial E^a} = I_a, \quad d\Phi = I_a dE^a. \quad (3)$$

We notice that equation $\Phi(E^a)$ must be explicitly given and it corresponds to the fundamental equation in standard thermodynamics.

Applying the pullback φ^* to the metric G , we get the induced metric g by $\varphi^*(G) = g$ resulting in

$$g^{GTD} = \varphi^*(G) = \left(E^c \frac{\partial \Phi}{\partial E^c}\right) \left(\eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d\right). \quad (4)$$

We can see that it is only necessary to know the fundamental equation $\Phi(E^a)$ to determine explicitly the metric g on \mathcal{E} . According to the GTD prescription, once we have $\Phi(E^a)$, we have all the requirements to associate the thermodynamics of the physical system to this geometrical structure. In particular, our principal interest is to obtain the phase transitions of the system which should correspond to singular points in the curvature scalar R as proposed in GTD [16]. This procedure to obtain the phase transitions of black holes systems with the GTD approach has been used successfully in previous works [17–23].

III. THERMODYNAMICS

The corresponding line element for the black hole with power Maxwell invariant (PMI) source [24] solution is,

$$ds^2 = -f(r)dt^2 + \frac{dr}{f(r)} + r^2 d\Omega_{d-2}^2, \quad (5)$$

where $d\Omega_{d-2}^2$ stands for standard element on S^d , with lapse function,

$$f(r) = 1 + \frac{r_+^2}{l^2} - \frac{m}{r_+^{n-2}} + \frac{(2s-1)^2 \left[\frac{(n-1)(2s-n)^2 q^2}{(n-2)(2s-1)^2} \right]^s}{(n-1)(n-2s)r_+^{\frac{2(ns-3s+1)}{2s-1}}}. \quad (6)$$

Here m and q are related with the ADM mass M and the electric charge Q by means of the relation,

$$m = \frac{16\pi M}{(n-1)\omega_{n-1}}, \quad (7)$$

$$q = \left[\frac{8\pi}{\sqrt{2s}\omega_{n-1}} \right]^{\frac{1}{2s-1}} \left[\frac{n-2}{n-1} \right]^{\frac{1}{2}} \frac{(2s-1)^{\frac{2s-2}{2s-1}}}{n-2s} Q^{\frac{1}{2s-1}}. \quad (8)$$

The horizons of the black hole with PMI source metric correspond to the roots of the lapse function $f(r)$. In terms of the exterior horizon radius r_+ , the black hole mass is given by the expression,

$$M(r_+, Q) = \frac{(n-1)\omega_{n-1}}{16\pi} \left[r_+^{n-2} + \frac{r_+^n}{l^2} - \frac{(2s-1)^{2-2s}(n-1)^{s-1}(2s-n)^{2s-1}}{(n-2)^s} r_+^{\frac{(2s-n)}{2s-1}} q^{2s} \right].$$

Furthermore, the entropy of the black hole with PMI source is defined as

$$S = \frac{\omega_{n-1} r_+^{n-1}}{4}, \quad (9)$$

In terms of this entropy, the corresponding thermodynamic fundamental equation is given by

$$M(S, Q) = \frac{(n-1)\omega_{n-1}}{16\pi} \left\{ \left[\frac{4S}{\omega_{n-1}} \right]^{\frac{n-2}{n-1}} + \left[\frac{4S}{\omega_{n-1}} \right]^{\frac{n}{n-1}} l^{-2} - \frac{(2s-1)^{2-2s}(n-1)^{s-1}(2s-n)^{2s-1}}{(n-2)^s} \left[\frac{4S}{\omega_{n-1}} \right]^{\frac{2s-n}{(n-1)(2s-1)}} q^{2s} \right\}, \quad (10)$$

with $\omega_{n-1} = (2\pi^{n/2})/\Gamma(n/2)$. In order to avoid inconsistent results, we will consider that the parameter s satisfies the relationships $2s-1 = i$ and $2s-n = i-n+1$, with i any integer and $n \neq i+1$ in order to keep real all terms in equation (9). The physical parameters of the black hole with PMI source satisfy the first law of black hole thermodynamics [25],

$$dM = TdS + \Phi dQ, \quad (11)$$

where T is the Hawking temperature which is proportional to the surface gravity on the horizon and Φ the electrical potential. As in ordinary thermodynamics, all the thermodynamic information is contained in the fundamental equation. The standard conditions of the thermodynamic equilibrium are given by the expressions,

$$T = \frac{\partial M}{\partial S}, \quad \Phi = \frac{\partial M}{\partial Q}. \quad (12)$$

According with the Ehrenfest's classification [26] the phase transitions take place where the derivatives of the Gibbs free energy $G = T M - \Phi Q$ diverge, for example, the first-order phase transitions exhibit a discontinuity in the first derivative of the G with respect to some thermodynamic variable and second-order phase transitions exhibit discontinuity in a second derivative of G . Because the heat capacity at constant electric charge C_Q is a second derivative of the free energy G , it will be used to describe the second order phase transition structure of the system,

$$C_Q = -T \left(\frac{\partial^2 G}{\partial T^2} \right)_Q = T \left(\frac{\partial S}{\partial T} \right)_Q = \frac{\left(\frac{\partial M}{\partial S} \right)_Q}{\left(\frac{\partial^2 M}{\partial S^2} \right)_Q}. \quad (13)$$

Using the equation (10) it is possible to compute the heat capacity at constant electric charge. However, it is not possible to write this quantity in a compact form. Therefore, we will take particular configurations of the black hole and its behaviour using different values of the parameter s and n . As an example of this analysis we consider the values $s = \frac{5}{2}$ and $n = 4$, corresponding to $i = 4$. The C_Q has the form,

$$C_Q = \frac{6S \left[300S^{\frac{5}{4}} + 2 \cdot 2^{\frac{3}{8}} S^{\frac{3}{4}} \pi^{\frac{5}{4}} l^2 Q^{\frac{5}{4}} + 75 \cdot 2^{\frac{1}{3}} \pi^{\frac{4}{3}} l^2 S^{\frac{7}{12}} \right]}{600S^{\frac{5}{4}} - 11 \cdot 2^{\frac{3}{8}} 5^{\frac{3}{4}} \pi^{\frac{5}{4}} l^2 Q^{\frac{5}{4}} - 150 \cdot 2^{\frac{1}{3}} \pi^{\frac{4}{3}} l^2 S^{\frac{7}{12}}}. \quad (14)$$

The second order phase transitions take place at those points where the heat capacity diverges, i. e.

$$600S^{\frac{5}{4}} - 11 \cdot 2^{\frac{3}{8}} 5^{\frac{3}{4}} \pi^{\frac{5}{4}} l^2 Q^{\frac{5}{4}} - 150 \cdot 2^{\frac{1}{3}} \pi^{\frac{4}{3}} l^2 S^{\frac{7}{12}} = 0. \quad (15)$$

The behaviour of C_Q is depicted in figure 1. We can see that C_Q has points of divergence which tell us, according with Ehrenfest's theory, that there is a point where a second order phase transition can take place.

Using the conditions of the thermodynamic equilibrium (12), one obtains the temperature and electric potential of the black hole on the event horizon as

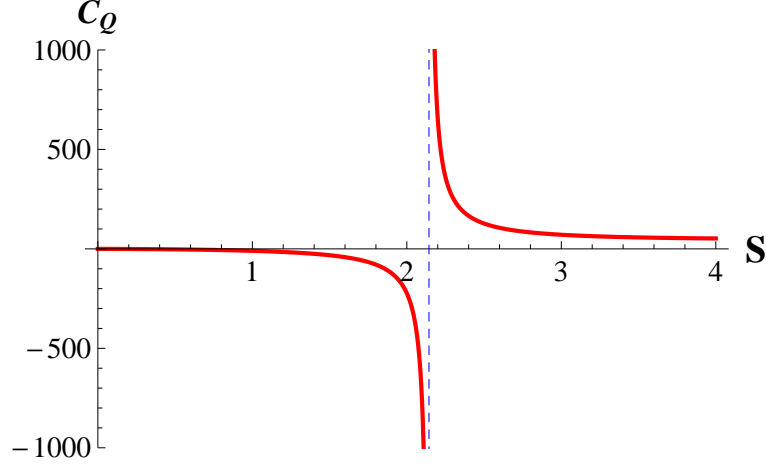


FIG. 1: The heat capacity C_Q as a function of the entropy S , for $l = 1$, $Q = 1$, $s = 5/2$ and $n = 4$.

$$T = \frac{\pi^{\frac{5}{3}}}{2^{\frac{1}{6}} S^{\frac{11}{12}} l^2} \left[300 S^{\frac{5}{4}} + 2 \cdot 2^{\frac{3}{8}} 5^{\frac{3}{4}} \pi^{\frac{5}{4}} l^2 Q^{\frac{5}{4}} + 75 \cdot 2^{\frac{1}{3}} \pi^{\frac{4}{3}} l^2 S^{\frac{7}{12}} \right], \quad (16)$$

$$\Phi = \frac{2^{\frac{17}{24}} 5^{\frac{3}{4}}}{10 \pi^{\frac{5}{12}}} S^{\frac{1}{12}} Q^{\frac{1}{4}}. \quad (17)$$

We see that all intensive thermodynamic variables are well-behaved. The general behaviour of these variables are illustrated in figure 2.

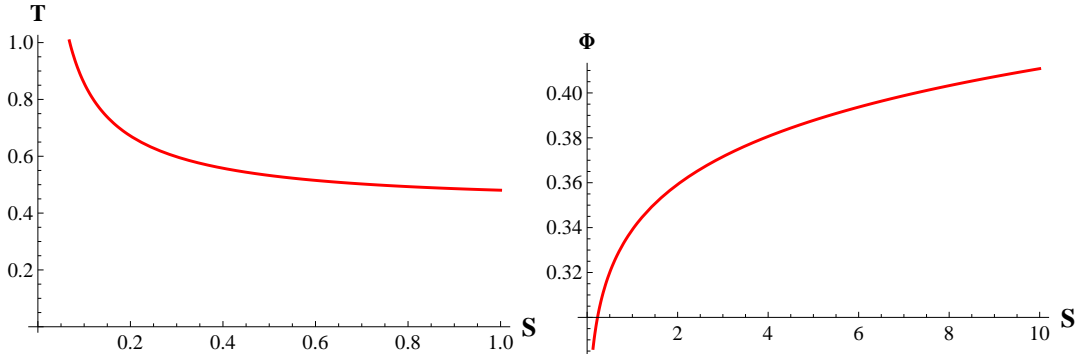


FIG. 2: The temperature T (left) and the electric potential Φ (right) as functions of the entropy S , for $l = 1$ and $Q = 1$. In these graphics we also consider: $s = 5/2$ and $n = 4$

In the same way we can analyze other examples, and in all of them we found a similar behaviour, i.e., there are points of divergence that corresponds to second-order phase transitions. Other examples are shown in the figure 3.

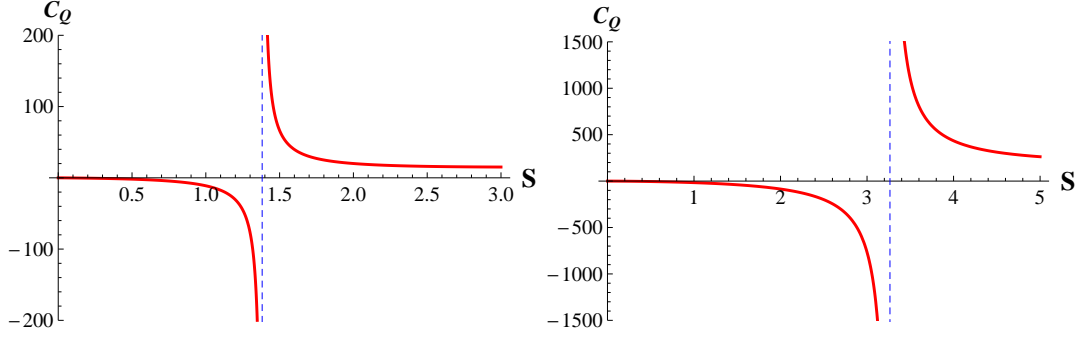


FIG. 3: The heat capacity C_Q as a function of the entropy S , for $l = 1$ and $Q = 1$. In these graphics we also consider: $s = 5/2$ and $n = 3$ (left), $n = 6$ (right)

IV. GEOMETROTHERMODYNAMICS OF THE BLACK HOLE WITH PMI SOURCE

The metric (4) with $\Phi = M$ and $E^a = \{S, Q\}$ reads,

$$g^{GTD} = \left(S \frac{\partial M}{\partial S} + Q \frac{\partial M}{\partial Q} \right) \left(- \frac{\partial^2 M}{\partial S^2} dS^2 + \frac{\partial^2 M}{\partial Q^2} dQ^2 \right). \quad (18)$$

Using the expressions for the M , as was given in Eq. (10), we compute the curvature scalar R^{GTD} corresponding to the metric (18). We found that the corresponding curvature scalar R^{GTD} is different to zero. This result indicates, in accordance with GTD, that in this black hole there exists thermodynamic interaction. We don't write the explicit form of the curvature scalar because it is a cumbersome expression that cannot be written in a compact form. However taking the same values used in the heat capacity (14), we can write the curvature scalar as,

$$R^{GTD} = \frac{\mathcal{F}(S, Q)}{\mathcal{P}_1^3 \mathcal{P}_2^2}, \quad (19)$$

with,

$$\mathcal{P}_1 = 300S^{\frac{5}{4}} + 32 \cdot 2^{\frac{3}{8}} 5^{\frac{3}{4}} \pi^{\frac{5}{4}} l^2 Q^{\frac{5}{4}} + 75 \cdot 2^{\frac{1}{3}} \pi^{\frac{4}{3}} l^2 S^{\frac{7}{12}}, \quad (20)$$

$$\mathcal{P}_2 = 600S^{\frac{5}{4}} - 11 \cdot 2^{\frac{3}{8}} 5^{\frac{3}{4}} \pi^{\frac{5}{4}} l^2 Q^{\frac{5}{4}} - 150 \cdot 2^{\frac{1}{3}} \pi^{\frac{4}{3}} l^2 S^{\frac{7}{12}}, \quad (21)$$

so the curvature singularities are determined by the zeros of the two polynomials entering the denominator. The function $\mathcal{F}(S, Q)$ is a polynomial that is different from zero at those points where the denominator vanishes. It follows that there exist curvature singularities at those points where the condition $600S^{\frac{5}{4}} - 11 \cdot 2^{\frac{3}{8}} 5^{\frac{3}{4}} \pi^{\frac{5}{4}} l^2 Q^{\frac{5}{4}} - 150 \cdot 2^{\frac{1}{3}} \pi^{\frac{4}{3}} l^2 S^{\frac{7}{12}} = 0$ is satisfied. These singularities coincide with the points where $C_Q \rightarrow \infty$, i. e., with the points where second order phase transitions take place, as given in equation (15). The general behaviour of the curvature scalar is illustrated in figure 4.

The behaviour of the curvature scalar, corresponding to the same values used in the heat capacities shown in figure 2, is shown in the figure 5.

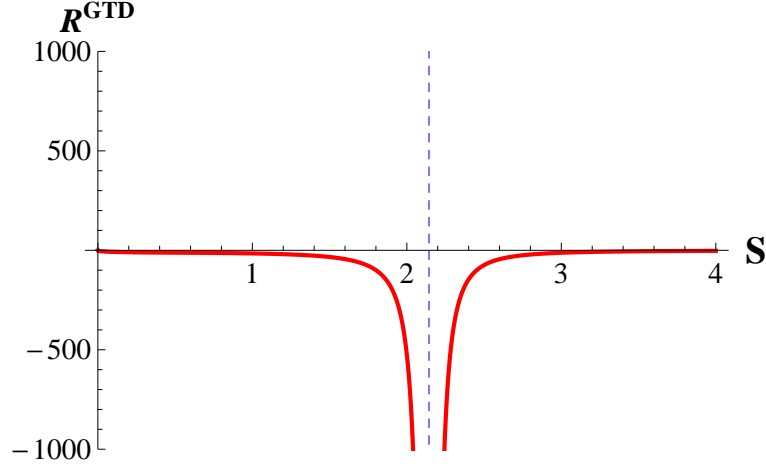


FIG. 4: The curvature scalar R^{GTD} as a function of the entropy S , with $l = 1$, $Q = 1$, $s = 5/2$ and $n = 4$.

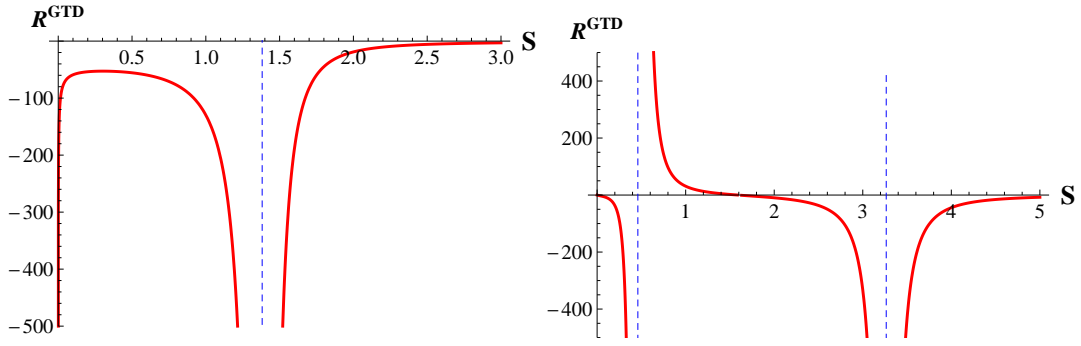


FIG. 5: The curvature scalar R^{GTD} as a function of the entropy S , for $l = 1$ and $Q = 1$. In these graphics we also consider: $s = 5/2$ and $n = 3$ (left), $n = 6$ (right)

As we can see the curvature scalar corresponding to $n = 6$ have two singularities; one of them, which is between zero and one, corresponds to $f = S \frac{\partial M}{\partial S} + Q \frac{\partial M}{\partial Q} = 0$ as is shown in figure 6; therefore, we should not consider it as a physical singularity. The other one, corresponds to the point where the capacity becomes infinite, which once again shows the correspondence between curvature singularities and second order phase transitions. The same happens for other examples

These results tell us, according with geometrothermodynamics, that there exist curvature singularities at those points where second order phase transitions occur [16].

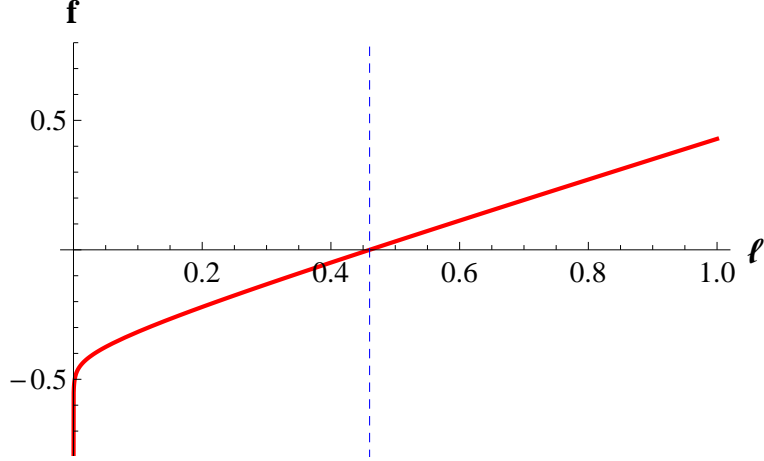


FIG. 6: The factor $f = S \frac{\partial M}{\partial S} + Q \frac{\partial M}{\partial Q}$ as a function of S , with $Q = 1$, $l = 1$, $s = 5/2$ and $n = 6$.

As a result we find all the intervals where the curvature scalar becomes singular correspond to divergent points of the heat capacity; so, geometrothermodynamics correctly describes the second order phase transitions of this thermodynamic system.

V. THE CASE $s = 1$

In $(n+1)$ -dimensions the PMI theory becomes conformally invariant, the same as Maxwell theory in four-dimensions, if the parameter has the value $s = (n+1)/4$ [8]. When we consider the particular case $s = 1$, we return to the Maxwell theory, i.e., a Reissner-Nordström black hole with linear electromagnetic source. In this section we will analyze this case in order to continue showing the consistency of GTD.

If $s = 1$, then n can be equal to three or larger. We choose $n = 3$ and the fundamental equation takes the form,

$$M(S, Q) = \frac{2\omega_2}{16\pi} \left\{ \left[\frac{4S}{\omega_2} \right]^{\frac{1}{2}} + \left[\frac{4S}{\omega_2} \right]^{\frac{3}{2}} l^{-2} + \left[\frac{4S}{\omega_2} \right]^{-\frac{1}{(2)}} q^2 \right\}, \quad (22)$$

with $\omega_2 = (2\pi^{3/2})/\Gamma(3/2)$ and $q = (8\pi Q)/\omega_2$. Using the equation (22) into the metric (18) we get the corresponding curvature scalar R^{GTD} ,

$$R^{GTD} = \frac{\mathcal{G}(S, Q)}{\left(l^2\pi S + 3S^2 + 3\pi^2 Q^2 l^2\right)^3 \left(3S^2 - \pi S l^2 + 3\pi^2 Q^2 l^2\right)^2}, \quad (23)$$

with,

$$\mathcal{G}(S, Q) = 48\pi^3 S^2 l^4 \left[\begin{aligned} &45S^5 - 6\pi^4 S Q^2 l^6 + 15\pi^3 S^2 Q^2 l^4 + 6\pi^5 Q^4 l^6 - 18\pi^2 S^3 Q^2 l^2 - \\ &- 63\pi^4 S Q^4 l^4 - 2\pi^2 S^3 l^4 - 3\pi S^4 l^2 \end{aligned} \right]. \quad (24)$$

As we can see $R^{GTD} \neq 0$ indicating thermodynamic interaction. The heat capacity at constant electric charge is computed by means of the equation (13):

$$C_Q = \frac{2S \left(3S^2 + \pi S l^2 - \pi^2 Q^2 l^2\right)}{3S^2 - \pi S l^2 + 3\pi^2 Q^2 l^2}. \quad (25)$$

We see from the expression for the curvature scalar that the singular points correspond to the values that fulfill the equation $3S^2 - \pi S l^2 + 3\pi^2 Q^2 l^2 = 0$ which are exactly the points where a phase transition occurs in the heat capacity (25). This behaviour is shown in figure 7.

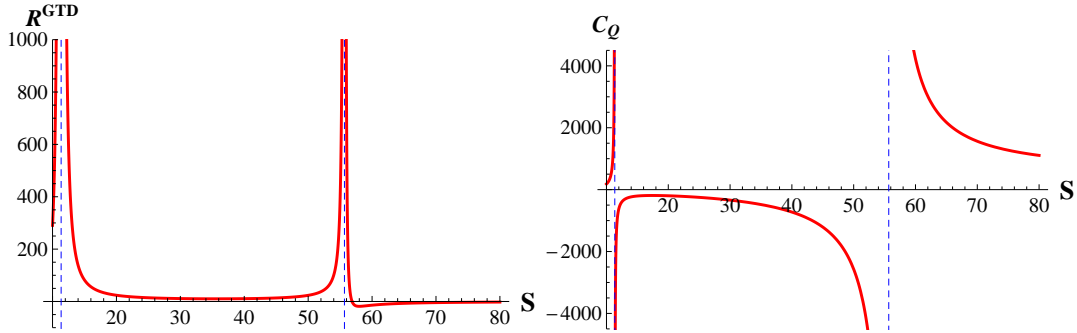


FIG. 7: Curvature scalar R^{GTD} (left) and heat capacity (right) as functions of S , for $Q = 1$, $l = 8$, $s = 1$ and $n = 3$.

We can notice that this RNAdS black hole has two points where heat capacity becomes singular and it is worthwhile to mention that these are the same points where the curvature scalar diverges. According with Ehrenfest classification and the GTD prescription this tell us that this black hole has two second order phase transition in agreement with [19, 27].

VI. THE WEINHOLD APPROACH

In this section we analyze the thermodynamic geometry of the black hole with PMI source by using the Weinhold metric which is defined as [11]

$$g^W = \frac{\partial^2 M}{\partial S^2} dS^2 + 2 \frac{\partial^2 M}{\partial S \partial Q} dS dQ + \frac{\partial^2 M}{\partial Q^2} dQ^2. \quad (26)$$

Using the mass M of the black hole with PMI source (10) the metric (26) can be calculated. The metric and corresponding scalar curvature are very complicated so we will not present them here. Instead we present one of the particular examples studied in the section (IV). Considering $s = \frac{5}{2}$ and $n = 4$ the curvatures scalar can be written as

$$R^W = \frac{1500 \cdot 2^{\frac{1}{12}} \cdot 5^{\frac{1}{2}} \pi^{\frac{9}{2}} S^2 l^4}{\left[15 \cdot 2^{\frac{3}{8}} \cdot 5^{\frac{3}{4}} \pi^{\frac{25}{12}} S^{\frac{4}{3}} l^2 - 60 \cdot 2^{\frac{1}{24}} \cdot 5^{\frac{3}{4}} \pi^{\frac{3}{4}} S^2 + 8 \cdot 2^{\frac{5}{12}} \cdot 5^{\frac{1}{2}} \pi^2 S^{\frac{3}{4}} Q^{\frac{5}{4}} l^2 \right]^2}, \quad (27)$$

The heat capacity at constant electric charge is given by the expression (13). We can see that the Weinhold geometry is curved, signaling interaction for this thermodynamic system. There are singular points, but they are not consistent with the ones of the heat capacity for fixed charge. In figure 8 we plot the scalar curvature and the heat capacity and in figure 9 we plot both functions in order to illustrate this result.

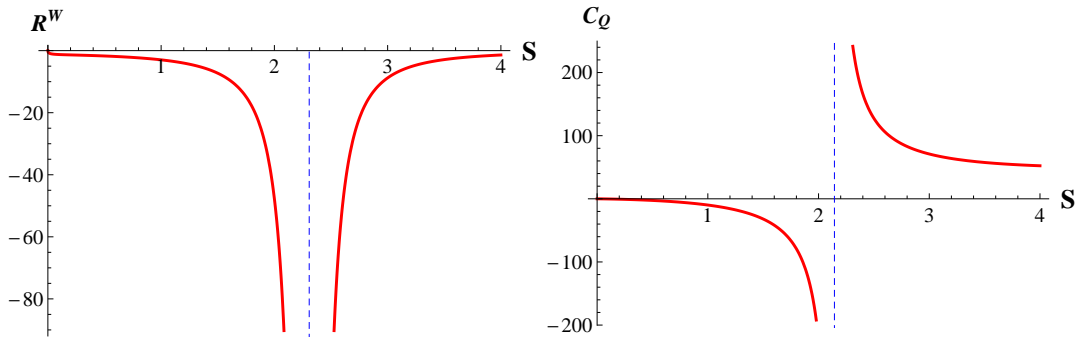


FIG. 8: The curvature scalar R^W (left) and the heat capacity (right) as functions of S , with $Q = 8$, $s = 5/2$ and $n = 4$.

The fact that this Weinhold curvature scalar doesn't diverge at the same points where the heat capacity does, tell us that it is not possible to associate curvature singularities with second-order phase transitions.

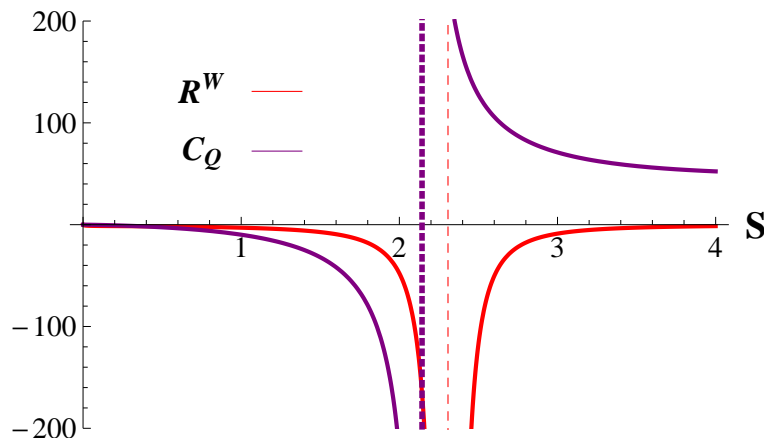


FIG. 9: The curvature scalar R^W (in red) and heat capacity (in purple) as functions of S , with $Q = 8$, $s = 5/2$ and $n = 4$. The dotted thick vertical line corresponds to the point where the heat capacity is singular and the dashed vertical line is where the curvature scalar of the Weinhold metric diverges.

The case of Ruppeiner's geometry for thermodynamical systems [12, 13] must be computed in the entropy representation, but this AdS black hole solution with nonlinear electromagnetic source doesn't allow us to write an explicit expression for the entropy in terms of the remaining variables. However, as has been shown in [28, 29], one can prove that Ruppeiner's metric, g^R , is proportional to Weinhold's metric, g^W as $g^R = (1/T)g^W$, where T is the temperature. This is why we expect that Ruppeiner's approach exhibits an equivalent behaviour as Weinhold's approach.

VII. COSMOLOGICAL CONSTANT AS A THERMODYNAMIC VARIABLE

Next, we will consider l as a thermodynamic variable [30]. The first law of thermodynamics can be written as

$$dM = TdS + \Phi dQ + Ldl, \quad (28)$$

where $L = \frac{\partial M}{\partial l}$ is the thermodynamic variable dual to l , the cosmological constant. Then, by means of relationship for the mass (10) we can obtain intensive thermodynamics variables T , Φ and L . Unfortunately, it is not possible to write these quantities in a simple form.

In order to solve this problem, as we have done in the section (III), we will consider some configurations of the black hole taking particular values for the different parameters. Taking the values: $s = \frac{5}{2}$ and $n = 4$, corresponding to $i = 4$, the C_Q has the form

$$C_Q = \frac{6 \left[75 \cdot 2^{\frac{1}{6}} \pi^{\frac{25}{12}} S^{\frac{7}{12}} l^2 + 150 \cdot 2^{\frac{5}{6}} \pi^{\frac{3}{4}} S^{\frac{5}{4}} + 2^{\frac{29}{24}} \cdot 5^{\frac{3}{4}} \pi^2 Q^{\frac{5}{4}} l^2 \right]}{300 \cdot 2^{\frac{5}{6}} \pi^{\frac{3}{4}} S^{\frac{5}{4}} - 11 \cdot 2^{\frac{5}{24}} \cdot 5^{\frac{3}{4}} \pi^2 Q^{\frac{5}{4}} l^2 - 150 \cdot 2^{\frac{1}{6}} \pi^{\frac{25}{12}} S^{\frac{7}{12}} l^2}. \quad (29)$$

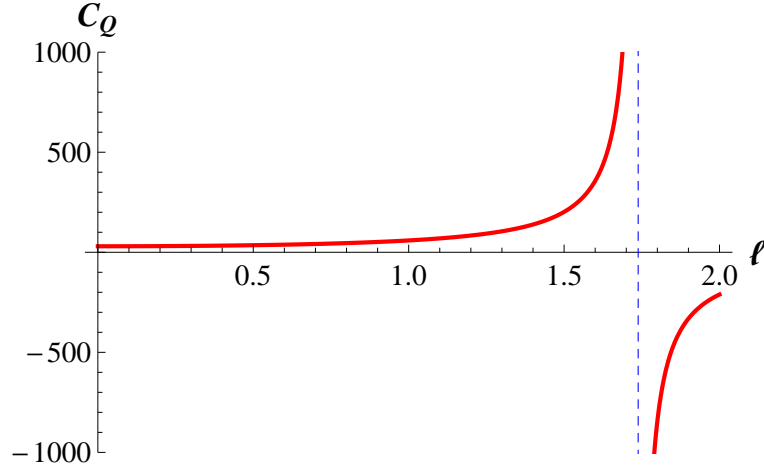


FIG. 10: The heat capacity C_Q as a function of l , for $Q = 1$, $S = 10$, $s = 5/2$ and $n = 4$.

The second order phase transitions take place at those points where the heat capacity diverges, i. e., for

$$300 \cdot 2^{\frac{5}{6}} \pi^{\frac{3}{4}} S^{\frac{5}{4}} - 11 \cdot 2^{\frac{5}{24}} \cdot 5^{\frac{3}{4}} \pi^2 Q^{\frac{5}{4}} l^2 - 150 \cdot 2^{\frac{1}{6}} \pi^{\frac{25}{12}} S^{\frac{7}{12}} l^2 = 0. \quad (30)$$

The behaviour of C_Q is depicted in figure 10. Using the conditions of the thermodynamic equilibrium we get the intensive thermodynamic variables:

$$T = \frac{2^{\frac{1}{2}}}{300 \pi^{\frac{29}{12}} S^{\frac{11}{12}} l^2} \left[75 \cdot 2^{\frac{1}{6}} \pi^{\frac{25}{12}} S^{\frac{7}{12}} l^2 + 150 \cdot 2^{\frac{5}{6}} \pi^{\frac{3}{4}} S^{\frac{5}{4}} + 2^{\frac{29}{24}} \cdot 5^{\frac{3}{4}} \pi^2 Q^{\frac{5}{4}} l^2 \right], \quad (31)$$

$$\Phi = \frac{2^{\frac{17}{24}} \cdot 5^{\frac{3}{4}}}{10 \pi^{\frac{5}{12}}} S^{\frac{1}{12}} Q^{\frac{1}{4}}, \quad (32)$$

$$L = -\frac{3 \cdot 2^{\frac{1}{3}} S^{\frac{4}{3}}}{2 \cdot \pi^{\frac{5}{3}} l^2}. \quad (33)$$

We see that all intensive thermodynamic variables are well-behaved. The general behaviour of these variables are illustrated in figure 11.

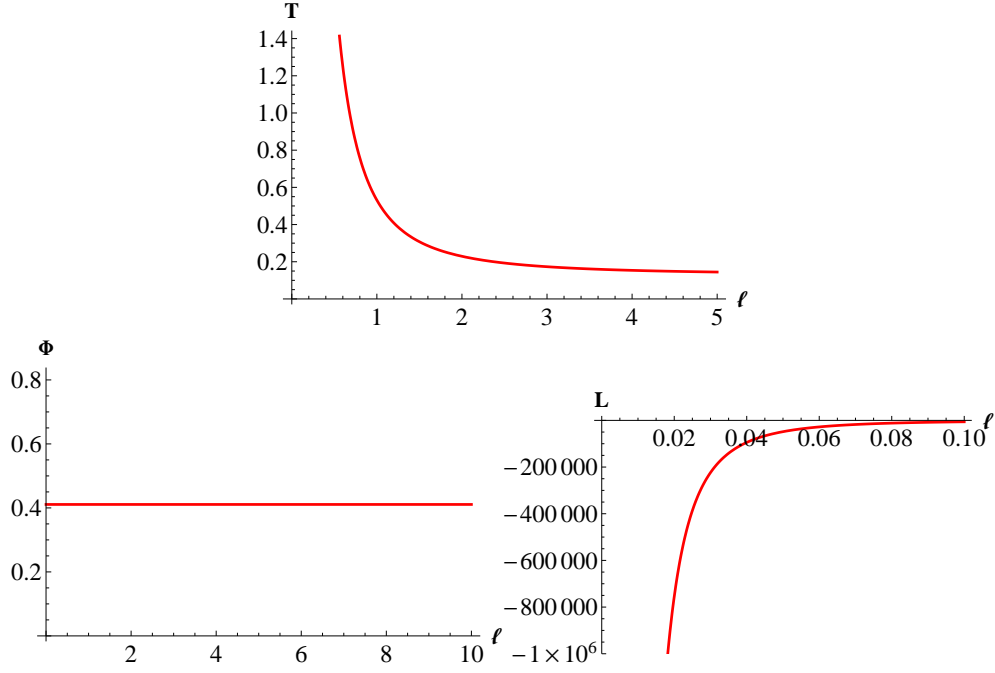


FIG. 11: T (left), Φ (center) and L (right) as functions of the l , for $Q = 1$, $S = 10$, $s = 5/2$ and $n = 4$.

Then, using the relationship for the induced metric g (4) with $\Phi = M$ and $E^a = \{S, Q, l\}$ we get,

$$g_I^{GTD} = \left(S \frac{\partial M}{\partial S} + Q \frac{\partial M}{\partial Q} + l \frac{\partial M}{\partial l} \right) \left(- \frac{\partial^2 M}{\partial S^2} dS^2 + \frac{\partial^2 M}{\partial Q^2} dQ^2 + 2 \frac{\partial^2 M}{\partial Q \partial l} dQ dl + \frac{\partial^2 M}{\partial l^2} dl^2 \right) \quad (34)$$

Inserting here the expression for the mass (10), we obtain a rather cumbersome metric which cannot be written in a compact form. Its corresponding scalar curvature turns out to be nonzero, i.e., according with GTD there is thermodynamic interaction. The explicit form of the curvature scalar cannot be written in a compact form but we can see its behaviour in figure 12.

From figure 12 we can see that R^{GTD} has two singularities but one of them corresponds to a point where the factor f of the metric g is zero, $f = S \frac{\partial M}{\partial S} + Q \frac{\partial M}{\partial Q} + l \frac{\partial M}{\partial l} = 0$. This

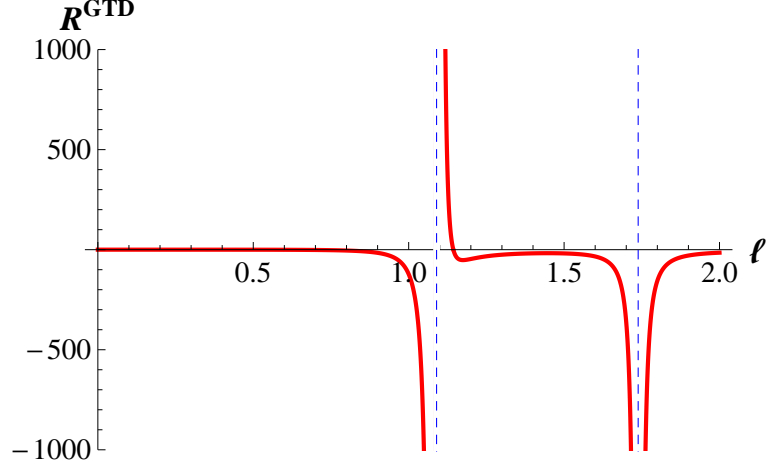


FIG. 12: The curvature scalar R^{GTD} as a function of l , with $Q = 1$, $S = 10$, $s = 5/2$ and $n = 4$.

can be seen in figure 13. This singular point is a singular point in the metric therefore it doesn't represent a physical divergence as the second singularity which takes place where the capacity becomes infinite. This second singularity describes a second order phase transition. Therefore, our analysis shows that the curvature singularities reproduce the structure of the phase transitions of a black hole with PMI source when we consider the cosmological constant as a thermodynamic variable.

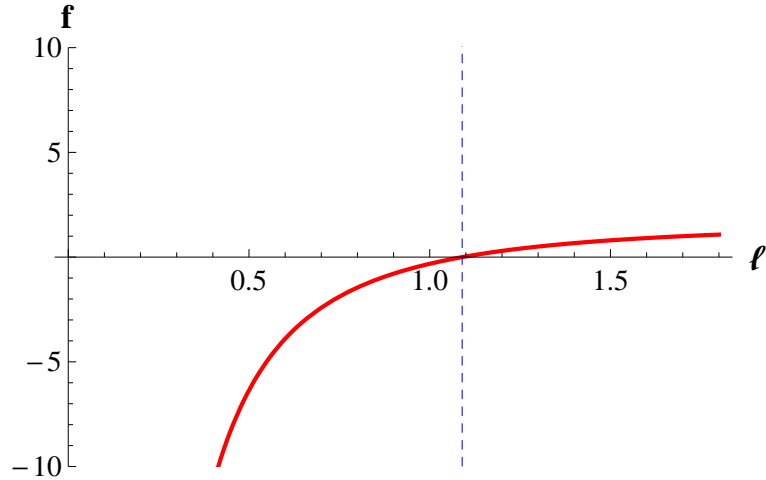


FIG. 13: The factor $f = S \frac{\partial M}{\partial S} + Q \frac{\partial M}{\partial Q} + l \frac{\partial M}{\partial l}$ as a function of l , with $Q = 1$, $S = 10$, $s = 5/2$ and $n = 4$.

VIII. CONCLUSIONS

In this paper, we analyzed the geometric structure of the equilibrium manifold of a black hole with PMI source. By means of the GTD formalism, which describes thermodynamic properties in terms of differential geometry in a Legendre invariant way, we derive the critical points that follow from the analysis of the divergences of the heat capacity. In black hole thermodynamics, the critical points of the heat capacity are usually associated with the occurrence of second order phase transitions. Here we analyzed the divergences of heat capacity and showed that GTD reproduces the behaviour of these critical points. We also studied the case $s = 1$ corresponding to the Reissner-Nordström and found that the PMI theory admits second phase transition which are situated at those points where the heat capacity diverges.

We analyzed the thermodynamic geometry based on the Weinhold metric and found that it is a curved manifold for the black hole with PMI source and the corresponding curvature diverges at some points, but these points are not the ones at which the heat capacity for fixed charge diverges. We conclude that Weinhold geometry does not describe correctly the thermodynamic geometry for the black hole with PMI source or at least it is not possible to give an interpretation of this divergences. We interpret this result as the need to impose the Legendre invariance in the context of the geometric analysis of the thermodynamics, since the Weinhold metric are not invariant with respect to Legendre transformations.

In the case where we consider the cosmological constant as a thermodynamic variable, we found that a curvature singularity appears exactly at that point where the phase transition occurs. Thus, GTD reproduces correctly the thermodynamic phase transition structure of a black hole with PMI source. These result reinforces the conclusion that GTD is able to correctly reproduce the phase transition structure of black holes.

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